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ANALYSIS OF FRICTIONLESS TWO-PHASE FLOW IN TUBES UNDER CENTRIFUGAL ACCELERATION AND MINIMUM HEATING FOR CHOKED FLOW†

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Abstract—To determine the void fraction in a tube of a rotating heat exchanger, an analytical investigation was undertaken to model frictionless two-phase flow boiling. Steady, one-dimensional separated two-phase conservation equations in differential form, were first applied to a stationary system. The equations were integrated between the inlet and exit of the flow channel to yield three coupled algebraic equations. The algebraic equations were then modified to represent rotating systems. To obtain closure, the velocity ratio, mass quality and void fraction are defined as a function of pressure.

A numerical technique was used to solve the equations. Sample results are presented in a graph of mass quality versus void fraction. The graph demonstrates that a minimum heat input must be exceeded to change from a single-phase flow to saturated two-phase flow boiling. Also, the void fraction was found to increase for increasing heat input, decreasing mass flow rate, increasing inlet mass quality and decreasing pressure difference between the inlet and exit.

INTRODUCTION

A rotating heat exchanger has been introduced by Leidenfrost & Eisele (1972) as an effective means to improve the performance of heat pumps. In this application tangential fans, utilizing airfoil shaped tubes in the impeller assembly, form the evaporator and condenser. As illustrated in figure 1, the working fluid is pumped through the hollow tubes as the unit rotates, about a central parallel axis, in the surrounding media.

The airfoil shaped blades of this system allow high velocities relative to the external fluid while maintaining a low absolute fluid velocity. Therefore, it is possible to attain high external heat transfer coefficients without considerably increasing the energy consumption. In fact, the thermal resistance depends to a large degree on the internal boiling or condensing heat transfer coefficient. This means information regarding boiling and condensing heat transfer coefficients in airfoil shaped tubes rotating about a parallel axis must be known to design an optimum rotary heat pump.

To ascertain saturated boiling heat transfer coefficients from experimental data obtained by White (1976), the surface area wetted by the liquid must be known. In most instances this area cannot be found directly, but indirectly through the void fraction. Since the void fraction appears in two-phase flow boiling equations, an analytical study was undertaken.

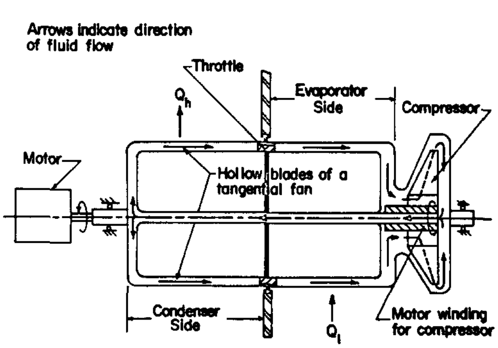

Figure 1. Schematic of rotary heat pump.

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Existing theories by Levy (1960, 1967), Zuber & Findlay (1965) and Kroeger & Zuber (1968) usually relate the void fraction to an indirect function of mass quality or drift flux, which is difficult to determine. In addition, many theories have been developed by assuming adiabatic flow, which certainly does not apply in a situation where boiling occurs. Although techniques are available such as described by Delhaye (1972) which enable one to measure these variables, it is more desirable to have an expression which only involves system parameters that can be easily determined or specified.

The rotational aspect of the system plays an important role in setting up the mathematical model. Inside the hollow blade, the centrifugal acceleration will force the liquid phase of the two-phase mixture outwards against the blade wall, as shown in figure 2. The vapor must then occupy the innermost portion of the blade. This means that the two-phase flow is separated and a unique liquid–vapor interface exists. Therefore, a separated two-phase flow model can be accurately applied to the rotary heat exchanger.

The succeeding paragraphs describe how one-dimensional two-phase conservation equations can be integrated to yield three algebraic expressions containing three unknowns (detailed information can be found in the Ph.D. thesis of White 1978). The analysis is first performed for stationary systems. The resulting equations are then modified to model the rotating system. A numerical solution for frictionless two-phase flow is also presented.

INTEGRATION OF CONSERVATION EQUATIONS

The conservation equations for one-dimensional, steady-state two-phase flow, restricted to thermodynamic equilibrium between the phases, are:

(i) Continuity

$$\frac{\mathrm{d}}{\mathrm{d}z}[\alpha\rho_G u_G + (1-\alpha)\rho_L u_L] = 0; \qquad [1]$$

where α is the void fraction, ρ the density and u the velocity. The subscripts G and L refer to vapor and liquid, respectively.

(ii) Momentum

$$\frac{\mathrm{d}}{\mathrm{d}z}[\alpha\rho_G u_G^2 + (1-\alpha)\rho_L u_L^2] + \frac{\mathrm{d}P}{\mathrm{d}z} - \frac{\mathrm{d}\tau}{\mathrm{d}z} + [\alpha\rho_G + (1-\alpha)\rho_L]g\sin\theta = 0; \qquad [2]$$

where P is pressure, τ the two-phase friction factor, g acceleration of gravity and θ the angle of inclination.

(iii) Energy

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\alpha\rho_G u_G\left(e_G + Pv_G + \frac{u_G^2}{2} + gz\sin\theta\right)\right.$$

$$\left. + (1-\alpha)\rho_L u_L\left(e_L + Pv_L + \frac{u_L^2}{2} + gz\sin\theta\right)\right] = \frac{4Q''}{D_h}, \qquad [3]$$

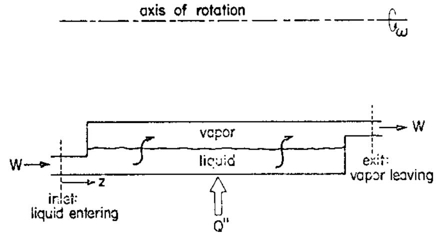

Figure 2. Schematic of hollow blade.

where e and v are specific internal energy and specific volume, Q'' is the heat flux and D_h the hydraulic diameter.

The continuity equation can be directly integrated to yield

$$\alpha\rho_G u_G + (1-\alpha)\rho_L u_L = G; \tag{4}$$

where the constant of integration, G, has a value equal to the total mass flux. The mass quality, x, and basic definitions are used in forming the equality

$$(1-\alpha)\rho_L u_L = (1-x)G. \tag{5}$$

Multiplying [4] by $(1-x)$ and substituting [5] produces

$$\epsilon = \frac{1-\alpha}{\alpha}\frac{\rho_L}{\rho_G}\frac{x}{1-x} \tag{6}$$

where ϵ is defined as the velocity ratio.

The momentum equation cannot be integrated in terms of an integration constant. Therefore, it is necessary to evaluate the equation as a definite integral. The momentum equation [2] can be rewritten in different form,

$$G\int_1^2 \mathrm{d}[xu_G + (1-x)u_L] + \int_1^2 \mathrm{d}P - \int_1^2 \left(\frac{\mathrm{d}\tau}{\mathrm{d}z}\right)\mathrm{d}z + \int_1^2 [\alpha\rho_G + (1-\alpha)\rho_L]g\sin\theta\,\mathrm{d}z = 0. \tag{7}$$

Equation [7] can be simplified by assuming that point 1 corresponds to the inlet of the flow channel, where conditions are known or can be determined easily, and point 2 refers to any position downstream. It can be shown that

$$[xu_G + (1-x)u_L]_1 = Gv_i, \tag{8}$$

where the specific volume at inlet state condition is

$$v_i = \frac{x_i^2}{\rho_{G_i}\alpha_i} + \frac{(1-x_i)^2}{\rho_{L_i}(1-\alpha_i)}. \tag{9}$$

Inserting [8] and [9] into [7] results in

$$G[xu_G + (1-x)u_L] - G^2 v_i + P - P_i - \int_i \left(\frac{\mathrm{d}\tau}{\mathrm{d}z}\right)\mathrm{d}z + \int_i [\alpha\rho_G + (1-\alpha)\rho_L]g\sin\theta\,\mathrm{d}z = 0. \tag{10}$$

Defining

$$R = \frac{P-P_i}{G^2} - \frac{1}{G^2}\int_i\left(\frac{\mathrm{d}\tau}{\mathrm{d}z}\right)\mathrm{d}z - v_i + \int_i [\alpha\rho_G + (1-\alpha)\rho_L]g\sin\theta\,\mathrm{d}z \tag{11}$$

and assuming that the liquid density remains approximately constant throughout the length of the flow channel, yields two dimensionless parameters designated as

$$\rho^* = \frac{\rho_L}{\rho_G} \qquad [12]$$

and

$$R^* = R\rho_L. \qquad [13]$$

Placing [11]–[13] into [10] and rearranging, through a series of manipulations, produces the algebraic form of the integrated momentum equation:

$$\epsilon^2 + \frac{(1-x)^2 + \rho^* x^2 + R^*}{x(1-x)}\epsilon + \rho^* = 0. \qquad [14]$$

The energy equation is integrated in the same fashion as the momentum equation. Recalling the definition of enthalpy,

$$i_G = e_G + Pv_G \qquad [15]$$

$$i_L = e_L + Pv_L \qquad [16]$$

and substituting [15] and [16] into [3], and then inserting [5] yields upon rearrangement

$$Gd\left[i_L + xi_{LG} + x\frac{u_G^2}{2} + (1-x)\frac{u_L^2}{2} + gz\sin\theta\right] = \frac{4Q''}{D_h}\,dz. \qquad [17]$$

Assuming that the heat flux remains constant throughout the length of the flow passage, [17] is integrated to yield

$$i_l^* + xi_{LG}^* + \rho^{*2}x^3 + 2\rho^* x^2(1-x)\epsilon + x(1-x)^2\epsilon^2$$

$$+\frac{\rho^{*2}x^2(1-x)}{\epsilon^2} + \frac{2\rho^* x(1-x)^2}{\epsilon} + (1-x)^3 - M^* = E^*; \qquad [18]$$

where

$$i_L^* = \frac{i_L}{1/2G^2}\rho_L \qquad [19]$$

$$i_{LG}^* = \frac{i_{LG}}{1/2G^2}\rho_L, \qquad [20]$$

$$M^* = \left[\frac{x_i^3}{\rho_{G_i}^2\alpha_i^2} + \frac{(1-x_i)^3}{\rho_L^2(1-\alpha_i)^2}\right]\rho_L^2, \qquad [21]$$

$$E^* = \frac{\rho_L^2}{1/2G^2}\left[\left(\frac{4Q''}{GD_h} - g\sin\theta\right)z + i_i\right]. \qquad [22]$$

Equation [18] is the algebraic form of the energy equation.

CHOKED FLOW CONDITIONS AND MAXIMUM MASS QUALITY

The conditions determining choked flow and maximum mass quality can be found from examination of the momentum and continuity equations. Eliminating the velocity ratio from these equations results in

$$x^2 + [(1-x)^2 + \rho^* x^2 + R^*]\frac{\eta}{\rho^*} + (1-x)^2\frac{\eta^2}{\rho^*} = 0\,, \qquad [23]$$

where

$$\eta = \frac{\alpha}{1-\alpha}. \qquad [24]$$

Since η defines a flow situation, it must be mathematically real; therefore, it can be shown, from the solution of [23], that the mass quality is restricted to the range

$$0 \le x \le \frac{1-\sqrt{(-R^*)}}{1-\sqrt{(\rho^*)}} \qquad [25]$$

in order to preserve the condition of real two-phase flow.

One can conclude from [25] that the maximum mass quality is given by

$$x_{max} = \frac{1-\sqrt{(-R^*)}}{1-\sqrt{(\rho^*)}}. \qquad [26]$$

The range of values for x_{max} is between zero and infinity. However, values greater than unity only indicate that it is possible to reach a state of superheated vapor.

Equation [26] is identical to the relationship derived by Richter (1971). The concept of maximum mass quality indicates that there is a maximum heat absorption capability for a two-phase flow, as previously verified by Bosnjakovic (1967).

Thc conditions determining choked flow can be found by examining the limits on the velocity ratio. Substituting the restriction given by [25] into the momentum equation [14], yields the limits of the velocity ratio for physically real two-phase flow, i.e.

$$-\sqrt{(\rho^*)} \le \epsilon \le \sqrt{(\rho^*)}. \qquad [27]$$

It is obvious from [27] that

$$|\epsilon_{max}| = \sqrt{(\rho^*)}. \qquad [28]$$

The above equation represents the condition for choked two-phase flow and occurs at maximum mass quality. Equation [28] is also known as Fauske's velocity ratio (see Fauske 1962, 1964). The previous relationship has been independently verified mathematically by Moody (1965) and Richter (1971).

INTEGRATED CONSERVATION EQUATIONS APPLIED TO ROTATING SYSTEMS

The conservation equations, as presented by [6], [14] and [18], have been derived for a stationary system. It is desirable to modify these equations such that they may be applied to rotating systems.

Two simplifications can be made regarding a system under rotation, provided that the centrifugal acceleration is much larger than the gravitation field:

(1) For the rotating system under consideration, the centrifugal acceleration becomes much

more significant than the gravitational field. Since the centrifugal acceleration is perpendicular to the flow coordinate z, the term containing $g \sin\theta$ can be eliminated due to $\sin\theta = 0$.

(2) As a result of the centrifugal forces, the void fraction remains constant along the length of the duct (this is also approximately true for vertical channels under 25 g's acceleration); therefore, $\alpha = \alpha_i = \alpha_0$. This assumption has been experimentally verified by White (1978). In the later instance α_0 is used to denote that the void fraction is not a function of length.

Utilizing the previous definition of void fraction, the continuity equation can be rewritten as

$$\epsilon = \frac{1-\alpha_0}{\alpha_0}\rho^*\frac{x}{1-x}. \qquad [29]$$

The form of the momentum and energy equation remain the same; however, the parameters R^*, M^* and E^* contained in these equations are changed similar to [29].

The conservation equations with the preceding modifications can now be used in studying two-phase flow in rotating ducts.

Examination of the conservation equations reveals that there are ten unknowns, ϵ, α_0, x, P, $(d\tau/dz)$, Q'', ρ^*, i_L^*, i_G^* and Δz (assuming that the inlet conditions and channel geometry are known). Two of these unknowns can be eliminated. The heat flux can be excluded by considering it as a condition or design parameter which is externally set. A few methods have been developed to predict the two-phase frictional pressure drop, e.g. Martinelli & Nelson (1948) and others. This means auxiliary equations to calculate $(d\tau/dz)$ are available, and hence, can be eliminated from the list of unknowns. The assumption of saturated flow allows the thermophysical properties to be related to pressure; therefore, they may also be eliminated. The continuity equation [29] can be used to eliminate the velocity ratio from the momentum and energy equations. This leaves two equations which involve the unknowns α_0, x, P and z. Since the number of unknowns exceeds the number of available equations, a unique state at a point z_0 cannot be mathematically determined.

Consider the diagram shown in figure 3. At a position z_1 a single value of α_0 and x does not exist, but an array of values dependent on the pressure P. The same can be said for positions z_2 and z_3. The rotational dynamics of the system require that α_0 be constant along the flow path (indicated by the dashed line). This means the pressure at points z_1, z_2 and z_3 are fixed by the physical conditions which must exict in a radial acceleration field. As demonstrated by figure 3, altering the pressure P_3 changes all pressures upstream such that the void fraction remains constant. It is also recognized that P_3 also defines a unique state for α_0 and x. By considering P to be the pressure at z, then P becomes an independent variable which can be used to determine α_0 and x. However, a relationship describing the pressure as a function of flow length, or vice versa, does not exist.

The last problem mentioned above can be partially overcome by looking at the boundary conditions. At the end of the flow channel the pressure obviously has a value equivalent to its exit value. Substituting $z = L$ and $P = P_e$ into the momentum and energy equations yields the mass quality and void fraction evaluated at the exit:

$$x_e^2[\rho_e^*(1-\alpha_0)+\alpha_0]+\alpha_0]+x_e[-2\alpha_0]+\alpha_0[1+(1-\alpha_0)R_e^*]=0, \qquad [30]$$

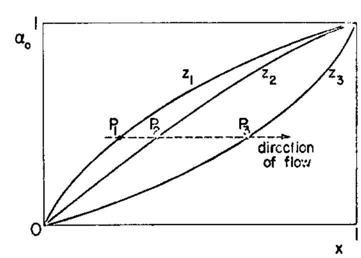

Figure 3. Qualitative representation of void ratio as a function of mass quality in a centrifugal system.

and

$$i^*_{Le} + x_e i^*_{LGe} + \frac{\Delta[\rho^{*2}x^3]_e}{\alpha_0^2} + \frac{\Delta[1-x)^3]_e}{(1-\alpha_0)^2} - E^*_e = 0\,, \qquad [31]$$

where

$$x_e = x(P_e), \quad \alpha_0 = \alpha_0(P_e)\,,$$

$$R^*_e = \frac{P_e - P_i}{G^2} - \frac{1}{G^2}\int_0^L \left(\frac{\mathrm{d}\tau}{\mathrm{d}z}\right)\mathrm{d}z - v^*_i\,, \qquad [32]$$

$$\Delta[\rho^{*2}x^3]_e = (\rho^{*2}x^3)_e - (\rho^{*2}x^3)_i\,, \qquad [33]$$

$$\Delta[(1-x)^3]_e = (1-x)_e^3 - (1-x)_i^3\,, \qquad [34]$$

$$E^*_e = \frac{\rho_L{}^2}{1/2G^2}\left[\frac{4Q''}{GD_h}L + i_i\right]. \qquad [35]$$

In [30] and [31] there are two unknowns, $\alpha_0(P_e)$ and $x_e(P_e)$; therefore, closure is possible and a unique solution exists.

SOLUTION FOR FRICTIONLESS FLOW

The simplest case to examine is that of frictionless flow. In this instance the problems involved in calculating the two-phase frictional pressure drop are eliminated.

By using the quadratic formula, x_e can be found as a function of void fraction and exit pressure from [30], i.e.

$$x_e = \frac{\alpha_0 + \sqrt{(-\alpha_0(1-\alpha_0)\{\rho^*_e + [\rho^*_e(1-\alpha_0)+\alpha_0]R^*_e\})}}{\rho^*_e(1-\alpha_0)+\alpha_0} \qquad [36]$$

Inserting the above equation into [31] results in a fifth degree polynomial in α_0, which cannot be solved exactly due to $\sqrt{(\alpha_0)}$ terms. However, a numerical solution is possible.

A simple trial and error procedure can be used to numerically solve [31] and [36]. The first step is to define the inlet conditions, channel geometry, heat absorption and exit pressure. The second step is to calculate ρ^*'s, i^*_L's and i^*_{LG}'s from thermophysical property relationships using the inlet and exit pressure. Next, in starting the trial and error procedure, a value for α_0 must be assumed to calculate a value of exit quality from [36]. The α_0 value chosen and the calculated value of x_e yield the heat absorption from [31]. Comparing the calculated value of heat absorption with the given value will indicate in which direction α_0 should be modified. For example, if $Q(cal) < Q(given)$, then the void fraction should be increased. The process of calculating and comparing heat absorption is repeated until the desired accuracy, between the calculated and given values, is reached. After the proper value of α_0 has been determined, the maximum mass quality and velocity ratio can be computed.

The numerical scheme proviously described was used to generate the plot shown in figure 4. The graph was computed using thermophysical properties of Refrigerant-11 and independent flow parameters picked at random.

RESULTS AND CONCLUSIONS

The most surprising result of figure 4 is the realization of a minimum heat input for two-phase flow. This suggests that for flow situations where the heat input is less than

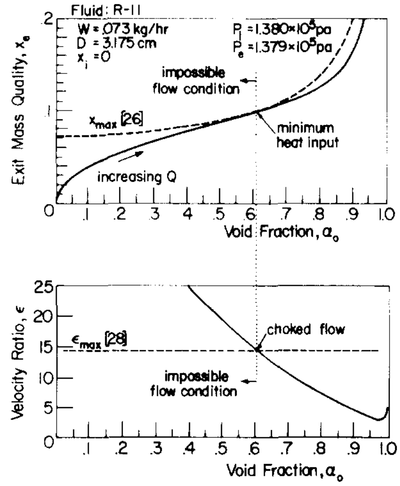

Figure 4. Exit mass quality and velocity ratio vs void fraction under frictionless flow in a rotating tube.

minimum, two-phase flow conditions do not exist but rather single-phase forced convection prevails. Considering that it has been experimentally proven by White (1977) that a few refrigerants (and possibly other fluids) have the ability to form superheated liquids, makes this explanation plausible.

From figure 4 it is also apparent that the flow chokes at minimum heat input. This is caused by the rapid transition from forced convection of the superheated liquid to two-phase flow. When the liquid changes to a vapor, the density changes on the order of 1000. This results in an instantaneous acceleration of vapor which causes the flow to choke.† Increasing the heat addition changes the upstream conditions such that the value of minimum heat input is exceeded. The liquid will then reach a level within the tube to achieve thermodynamic equilibrium. This phenomenon has been previously observed in earlier experiments by White (1976). However, at this time and before the present analysis it was not realized what was taking place and an experimental error was attributed to the observation. It should also be mentioned that the flow again chokes when the maximum heat absorption is reached for values of void fraction close to unity (see also Bosnjakovic 1967). The conditions of minimum and maximum heat absorption are unstable. In order to acheive a stable state, the flow will either adjust the inlet or exit conditions of the flow channel. If the flow did not adjust itself, boiling and generating superheated vapor would be impossible.

Plots similar to figure 4 can be generated by changing the independent flow parameters; and even through these flow conditions are arbitrary, the trends are always the same. There is a value of input heat which must be exceeded to change from single-phase forced convection to two-phase saturated boiling. Changing the independent flow variables also indicates that, in general, the void fraction increases for increasing heat absorption, decreasing mass flow rate, increasing inlet mass quality and decreasing pressure difference between the inlet and exit.

NOMENCLATURE

D_h hydraulic diameter,
e internal energy per unit mass, kJ/kg
G total mass flux, $kg/m^2 \cdot s$
g gravitational constant, $9.8\ m/s^2$

†Air to air heat pumps are presently considered for more efficient heating. Ice formation on the evaporator might lower the heat input until the minimum value is reached. The system will then fail to operate.

i specific enthalpy, kJ/kg
i_{LG} $i_G - i_L$, latent heat of vaporization, kJ/kg
L flow channel length, m
P pressure, Pa
Q'' heat flux, W/m^2
u velocity, m/s
v specific volume, m^3/kg
x mass quality
z flow direction, coordinate, m
α void fraction
α_0 void fraction for rotating systems
ϵ velocity ratio
ρ density, kg/m^3
τ two-phase frictional scalar, Pa

Subscripts
L refers to liquid
G refers to vapor
i refers to inlet state

REFERENCES

Bosnjakoric, F. 1967 Thermodynamische Einschränkungen bei beheizten Rohrströmumgen. *Wissenschaftliche Zeitschrift der Technischen Universität Dresden* **16**, 331–341.

Delhaye, J. M., Semeria, R. & Flamand, C. 1972 Void fraction, vapor and liquid temperatures local measurements in two-phase flow using a microthermocouple. ASME 72-HT-13.

Fauske, H. K. 1962 Contribution to the theory of two-phase, one component critical flow. USAEC Rep. ANL-6633.

Fauske, H. K. 1964 The discharge of saturated water through tubes. A.I.Ch.E. Preprint 30, *7th Natl Heat Transfer Conf.*

Kroeger, P. G. & Zuber, N. 1968 An analysis of the effects of various parameters on the average void fractions in subcooled boiling. *Int. J. Heat and Mass Transfer* **11**, 211–233.

Leidenfrost, W. & Eisele, H. E. 1972, Rotating heat exchangers and the optimization of a heat pump. *IEEE Trans. on Industry Appl.* **3**, 345–355.

Levy, S. 1960 Steam slip-theoretical prediction from momentum model. *J. Heat Transfer, ASME Trans.* **82**, 113–124.

Levy, S. 1967 Forced convection subcooled boiling-prediction of vapor volumetric fraction. *Int. J. Heat and Mass Transfer* **10**, 951–965.

Martinelli, R. C. & Nelson, D. B. 1948 Prediction of pressure deops during forced circulation boiling of water. *ASME Trans.* **70**, 695–702.

Moody, F. J. 1965 Maximum flow rate of a single component, two-phase mixture. *J. Heat Transfer, ASME Trans.* **87**, 134–142.

Richter, H. 1971 Zur Thermodynamik Verdampfender Zwei-phasen Rohrströmungen. Ph.D. Thesis, University of Stuttgart, Germany.

Wallis, G. B. 1969 *One-Dimensional Two-Phase Flow.* McGraw-Hill, New York.

White, R. D. 1976 Experimental and analytical analysis of boiling heat transfer under high centrifugal acceleration. M. S. Thesis, School of Mechanical Engineering, Purdue University, West Lafayette, Indiana.

White, R. D., Ku, J. T. & Leidenfrost, W. 1976 Boiling transfer coefficient of refrigerants flowing inside airfoil shaped tubes rotating about a parallel axis. School of Mechanical Engineering, Herrick Laboratories Rep. 77–25.

WHITE, R. D. 1978 Determination of void fractions for improved boiling heat transfer evaluation in rotating heat exchangers. Ph.D. Thesis, School of Mechanical Engineering, Purdue University, West Lafayette, Indiana.

ZUBER, N. & FINDLAY, J. A. 1965 Average volumetric concentration in two-phase flow systems, *J. Heat Transfer, ASME Trans.* **10**, 951–965.